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Note

Approximation for the Turning Points of Bessel Functions

1. Introduction

The zeros and turning points of the Bessel functions $J_\nu(x)$ have important applications in mathematical physics, applied mathematics, and numerical analysis. We denote the sth positive zero of $J_\nu(x)$ $(\nu > -1)$ by $j_{\nu,s}$ and the sth positive turning point of $J_\nu(x)$ $(\nu > 0)$ by $j'_{\nu,s}$, i.e., $j'_{\nu,s}$ is the sth positive zero of $J'_\nu(x)$. For $\nu = -1$, $x = 0$ is counted as the first positive zero of $J_{-1}(x)$. For $\nu = 0$, $x = 0$ is counted as the first positive turning point of $J_0(x)$. For fixed ν and large s, $j_{\nu,s}$ and $j'_{\nu,s}$ can be approximated accurately by using McMahon's asymptotic expansion [5]. For large ν, Olver's uniform asymptotic expansions [4] are very useful. Unfortunately, for constructing Fourier–Bessel series and for many other applications, we need $j_{\nu,s}$ for s and ν small. Recently, Piessens has given a power series expansion for $j_{\nu,1}$, which is convergent for $\nu \in [-1, 0)$ (see [7]) and Chebyshev series approximations for $j_{\nu,s}$, $-1 \leqslant \nu \leqslant 5$ and $s = 1, 2, 3, 4, 5$, and 6 (see [6]).

The purpose of this note is to present similar series approximations for $j'_{\nu,s}$.

2. Power Series Expansions for $j'_{\nu,1}$

In [1] Cayley has given an algorithm for the numerical calculation of zeros of oscillating functions, defined by a power series. We give here a brief description of this algorithm.

Let

$$f(x) = \prod_{s=1}^{\infty} \left(1 - \frac{x}{p_s}\right) \tag{1}$$

be a function with an infinite number of positive zeros p_i, $i = 1, 2, \ldots$, where $0 < p_1 < p_2 < \cdots$. If the power series expansion of $f(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \, a_k^{(1)} x^k, \tag{2}$$

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then

$$a_1^{(l)} = \sum_{s=1}^{\infty} p_s^{-2^l}, \qquad l = 0, 1, 2,..., \tag{3}$$

where

$$a_k^{(l)} = 2 \sum_{j=0}^{k}{}' (-1)^j a_{k-j}^{(l-1)} a_{k+j}^{(l-1)}, \qquad l = 1, 2, 3,..., k = 0, 1, 2,.... \tag{4}$$

The notation $\sum'$ is used to indicate that the first term in the summation must be halved. Formula (3) is useful for computing approximations for p_1.

Applying this algorithm to the power series expansion of $J'_\nu(x)$, we can compute

$$\sigma_r(\nu) = 2^r \nu^r (\nu+1)^r \sum_{s=1}^{\infty} (j'_{\nu,s})^{-2r}, \qquad r = 1, 2,.... \tag{5}$$

The results are

$$\sigma_1(\nu) = \nu + 2,$$

$$\sigma_2(\nu) = \frac{\nu^2 + 8\nu + 8}{4(\nu+2)},$$

$$\sigma_4(\nu) = \frac{5\nu^5 + 139\nu^4 + 800\nu^3 + 1888\nu^2 + 1984\nu + 768}{16(\nu+2)^2 (\nu+3)(\nu+4)},$$

$$\begin{aligned}\sigma_8(\nu) = {} & (429\nu^{13} + 40408\nu^{12} + 1065680\nu^{11} + 14483542\nu^{10} + 122067803\nu^9 \\ & + 691722994\nu^8 + 2745117824\nu^7 + 7776543488\nu^6 + 15791614976\nu^5 \\ & + 22771068928\nu^4 + 22717898752\nu^3 + 14881456128\nu^2 \\ & + 5748424704\nu + 990904320) \\ & : [256(\nu+1)^4 (\nu+3)^2 (\nu+4)^2 (\nu+5)(\nu+6)(\nu+7)(\nu+8)].\end{aligned}$$

Consequently, since $j'_{\nu,1} < j'_{\nu,2} < \cdots$, we have

$$j'_{\nu,1} = \sqrt{2\nu(\nu+1)} \lim_{r\to\infty} \phi_r(\nu), \tag{6}$$

where

$$\phi_r(\nu) = [\sigma_r(\nu)]^{-1/2r}. \tag{7}$$

Indeed, the effect of the limiting process in (6) is to pull out the dominant term of the sum in (5). Since $\sigma^{(r)}(0) = 1$, $r = 1, 2, 3,...$, we have the following asymptotic equivalence

$$j'_{\nu,1} \sim \sqrt{2\nu(\nu+1)}, \qquad \nu \to 0. \tag{8}$$

By approximating $\phi_r(v)$ by a Taylor polynomial, we obtain

$$j'_{v,1} = \sqrt{2v(v+1)} \sum_{k=0}^{r-1} c_k v^k + o(v^{r-1}), \qquad v \to 0, \tag{9}$$

where

$$c_k = \frac{1}{k!} \frac{d^k}{dv^k} \phi_r(v) \bigg|_{v=0}. \tag{10}$$

When $r \to \infty$, (10) becomes a power series expansion for $j_{v,1}$, which, because of the presence of a branch point of $\phi_r(v)$ at $v = -2$, converges only for $v < 2$.

We have evaluated c_k by symbolic differentiation using REDUCE [3], giving the following series expansion,

$$j'_{v,1} = \sqrt{2v(v+1)} \left[1 - \frac{1}{8} v + \frac{29}{384} v^2 - \frac{469}{9216} v^3 + \frac{160237}{4423680} v^4 - \cdots \right]. \tag{11}$$

This expansion can be transformed into

$$j'_{v,1} = \left[2v \left(1 + \frac{3}{4} v - \frac{1}{12} v^2 + \frac{53}{1152} v^3 - \frac{823}{27648} v^4 + \cdots \right) \right]^{1/2}. \tag{12}$$

Numerical experiments show that (12) is a more accurate approximation for $j'_{v,1}$, than (11). In Table I we give the exact values of $j'_{v,1}$ and the absolute errors of the five-terms approximations (11) and (12), for some values of v.

3. Chebyshev Series Approximations for $j'_{v,s}$

The power series expansions (11) and (12) have rather a theoretical interest because of their slow convergence. Therefore we present here Chebyshev series

TABLE I

		Absolute error of the five-terms approximation	
v	Exact $j'_{v,1}$	(11)	(12)
0.25	0.769062	−0.2(−4)	0.6(−5)
0.5	1.165561	−0.7(−3)	0.2(−3)
0.75	1.514337	−0.7(−2)	0.2(−2)
1.0	1.841184	−0.3(−1)	0.7(−2)
1.25	2.155151	−0.1(0)	0.2(−1)
1.5	2.460536	−0.3(0)	0.5(−1)

Table 2
Chebyshev coefficients for $j'_{\nu,1}$ (formula(13))

k	$c_k^{(1)}$
0	3.146037908354
1	0.500882082686
2	-0.054749823677
3	0.012285153373
4	-0.003455517165
5	0.001084943438
6	-0.000363054912
7	0.000126546152
8	-0.000045356977
9	0.000016586295
10	-0.000006156913
11	0.000002312057
12	-0.000000876227
13	0.000000334556
14	-0.000000128530
15	0.000000049637
16	-0.000000019256
17	0.000000007499
18	-0.000000002930
19	0.000000001149
20	-0.000000000452
21	0.000000000178
22	-0.000000000070
23	0.000000000028
24	-0.000000000011
25	0.000000000004
26	-0.000000000002
27	0.000000000001

Table 3
Chebyshev coefficients for $j'_{\nu,2}$ (formula(14),s=2)

k	$c_k^{(2)}$
0	14.552832352691
1	3.322944306693
2	-0.095175053289
3	0.019561278193
4	-0.004973483180
5	0.001415780934
6	-0.000432248049
7	0.000138360360
8	-0.000045811236
9	0.000015554459
10	-0.000005384163
11	0.000001892235
12	-0.000000673181
13	0.000000241897
14	-0.000000087648
15	0.000000031983
16	-0.000000011741
17	0.000000004333
18	-0.000000001606
19	0.000000000598
20	-0.000000000223
21	0.000000000084
22	-0.000000000031
23	0.000000000012
24	-0.000000000004
25	0.000000000002
26	-0.000000000001

Table 4
Chebyshev coefficients for $j'_{\nu,3}$ (formula(14),s=3)

k	$c_k^{(3)}$
0	21.169334382429
1	3.473357431327
2	-0.080988347543
3	0.011972168797
4	-0.002187249295
5	0.000447342551
6	-0.000098189752
7	0.000022623114
8	-0.000005399801
9	0.000001323869
10	-0.000000331451
11	0.000000084388
12	-0.000000021781
13	0.000000005686
14	-0.000000001498
15	0.000000000398
16	-0.000000000106
17	0.000000000029
18	-0.000000000008
19	0.000000000002
20	-0.000000000001

Table 5
Chebyshev coefficients for $j'_{\nu,4}$ (formula(14),s=4)

k	$c_k^{(4)}$
0	27.628911761509
1	3.561263085309
2	-0.070073551867
3	0.008224377211
4	-0.001192174007
5	0.000193337658
6	-0.000033635119
7	0.000006141076
8	-0.000001161550
9	0.000000225708
10	-0.000000044802
11	0.000000009047
12	-0.000000001853
13	0.000000000384
14	-0.000000000080
15	0.000000000017
16	-0.000000000004
17	0.000000000001

Table 6
Chebyshev coefficients for $j'_{\nu,5}$ (formula(14),s=5)

k	$c_k^{(5)}$
0	34.023909800050
1	3.619781378284
2	-0.061608527718
3	0.006029244440
4	-0.000728527495
5	0.000098444257
6	-0.000014265063
7	0.000002168734
8	-0.000000341502
9	0.000000055239
10	-0.000000009127
11	0.000000001534
12	-0.000000000262
13	0.000000000045
14	-0.000000000008
15	0.000000000001

Table 7
Chebyshev coefficients for $j'_{\nu,6}$ (formula(14),s=6)

k	$c_k^{(6)}$
0	40.385010455750
1	3.661799350703
2	-0.054912589226
3	0.004619722605
4	-0.000479813779
5	0.000055716435
6	-0.000006936189
7	0.000000905748
8	-0.000000122480
9	0.000000017011
10	-0.000000002413
11	0.000000000348
12	-0.000000000051
13	0.000000000008
14	-0.000000000001

expansions for $j'_{\nu,s}$ for the region $0 \leqslant \nu \leqslant 5$, $s = 1, 2, 3, 4, 5$, and 6. These new expansions, together with McMahon's asymptotic expansion for $s \to \infty$, and Olver's uniform asymptotic expansion for $\nu \to \infty$, allow the computation of $j'_{\nu,1}$ to at least 10 decimal figures for all $s \geqslant 1$ and $\nu \geqslant 0$. To take into account the asymptotic behaviour $j'_{\nu,1} \sim (2\nu)^{1/2}$, $\nu \to 0$, $j'_{\nu,1}$ is approximated by

$$j'_{\nu,1} \simeq (2\nu)^{1/2} \sum_{k=0}^{N_1}{}' c_k^{(1)} T_k^*\left(\frac{\nu}{5}\right), \qquad 0 \leqslant \nu \leqslant 5 \tag{13}$$

where $T_k^*(x)$ is the shifted Chebyshev polynomial and where the prime indicates that the first term is taken with factor $\frac{1}{2}$. For the following zeros we use the approximation

$$j'_{\nu,s} \simeq \sum_{k=0}^{N_s}{}' c_k^{(s)} T_k^*\left(\frac{\nu}{5}\right), \qquad 0 \leqslant \nu \leqslant 5, s = 2, 3, 4, 5, 6. \tag{14}$$

The coefficients $c_k^{(s)}$ are computed numerically using an algorithm proposed by Gentleman [2], and are presented in Tables II–VII.

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References

1. A. Cayley, *Proc. London Math. Soc.* **5** (1873–1874), 123.
2. W. M. Gentleman, *Commun. ACM* **15** (1972), 343.
3. A. C. Hearn, REDUCE 2–A System and Language for Algebraic Manipulation, *in* "Proceedings Second Symposium on Symbolic and Algebraic Manipulation, Los Angeles, 1971. 128–133.
4. F. W. J. Olver, *Philos. Trans. Roy. Soc. London Ser. A* **247** (1954), 328.
5. F. W. J. Olver, *in* "Royal Society Mathematical Tables, Bessel Functions," Part III, Vol. 7, Cambridge Univ. Press, London, 1960.
6. R. Piessens, *J. Comput. Phys.* **53** (1984), 188.
7. R. Piessens, *Math. Comput.* **42** (1984), 195.

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